

Cosmological perturbations in the presence of a solid with positive pressure

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Abstract

Evolution of scalar perturbations in a universe containing solid matter with positive pressure is studied. Solution for pure solid is found and matched with solution for ideal fluid, including the case when the pressure to energy density ratio w has a jump. Two classes of solutions are explored in detail, solutions with radiation-like solid ($w = 1/3$) and solutions with stiff solid ($w > 1/3$) appearing in a universe filled with radiation. For radiation-like solid, an almost flat spectrum of large-scale perturbations is obtained only if the shear stress to energy density ratio ξ is close to zero, $|\xi| \lesssim 10^{-5}$. For a solid with stiff equation of state, large-scale perturbations are enhanced for ξ negative and suppressed for ξ positive. If the solid dominated the dynamics of the universe long enough, perturbations could end up suppressed as much as by several orders of magnitude, and in order that the inclination of the large-scale spectrum is consistent with observations, radiation must have prevailed over the solid long enough before recombination. In Newtonian gauge, corrections to metric and energy density are typically much greater than 1 in the first period after the shear stress appears, but the linearized theory is still applicable because the corrections stay small when one uses the proper-time comoving gauge.

1 Introduction

One of possible modifications of the concordance Λ CDM model is adding solid matter with negative parameter w (pressure to energy density ratio) to the known components of the universe [1, 2]. Such matter can be composed of frustrated cosmic strings, which have $w = -1/3$, or domain walls,

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which have $w = -2/3$. The solid can, in principle, function as dark energy, making the universe accelerate with no unstable perturbations appearing in it. Acceleration of the universe, however, requires $w < -1/3$, so that it cannot be achieved with cosmic strings. Domain walls are of no help either, because the parameter w of the acceleration driving medium must be in absolute value greater than $2/3$ to reconcile observations, as established first for constant w [3] and then for w depending on time [4, 5, 6]. Moreover, domain walls in an expanding universe seem to be scaling rather than frustrated [7, 8], so that their number per *horizon*, rather than per unit comoving volume, is constant. This leads to an effective w which is in absolute value less than $2/3$. (In a radiation dominated universe it equals $-1/3$.) Nevertheless, scenarios containing solid matter with negative w continue to attract interest. The theory was extended to anisotropic solids [9, 10] and used as a general framework to describe the dark sector exclusively by means of metric tensor [11, 12]; effects of a lattice of cosmic strings superimposed on the conventional dark sector were studied [13]; and a possibility that solid matter could replace inflaton field as an inflation driving medium was contemplated [14, 15, 16, 17].

Existing literature on cosmology with solid matter apparently does not contain an analysis of the case with *positive* w . This case is obviously less interesting than that with negative w , which offers an alternative view of the least understood component of the known universe. The study of solids with positive w can be nevertheless useful, if only because it would extend the parametric space of the theory and give us greater freedom when explaining observational data.

The concordance model gives satisfactory explanation of observations, so that there is no need to add a new component to it. However, it is *possible* to do so and not to come into conflict with the data. The modified theory can yield the same power spectrum of CMB anisotropies as the original one if it assigns different values to cosmological parameters, or if it imposes different initial conditions on cosmological perturbations. In the former case, the theory could be verified in the future by alternative methods of measuring cosmological parameters, like determining h from the data on double quasars or from Zel'dovich-Sunyaev effect, if they become precise enough; in the latter case, the modified theory would pick a different set of inflationary scenarios than that preferred by the original theory, and could be verified by measuring quantities like tensor-to-scalar ratio sensitive to the details of inflation. To demonstrate these two possibilities, consider the two specific kinds of solids explored in the text. For a universe with both radiation-like and stiff solid, the theory predicts different behavior of large-scale anisotropies than for a universe with pure fluid, but this cannot be cured by changing the parameters of the concordance model; instead, we must tune the new parameters entering the theory. However, a radiation-like solid could influence significantly also the behavior of *short-scale* anisotropies, provided it has viscosity and thus produces an additional Silk dumping. The total dumping could be then suppressed by reducing the value of $\Omega_m h^2$. On the other hand, after we ensure that large-scale anisotropies

are the same on all scales, the presence of a stiff solid still affects their *size*, provided the initial size of cosmological perturbations is fixed. If the shear modulus is positive, the perturbations are *suppressed*, yielding smaller anisotropies than in a universe with pure fluid; thus, to accommodate observations the initial perturbations must be greater.

In order that a solid plays any role in the evolution of the universe, it must stay solid while being stretched by many orders of magnitude. “Stay solid” means that its shear modulus, or moduli if it is anisotropic, are nonzero and comparable with the pressure acting inside it. Solids with positive pressure can have effect on the dynamics of the universe during the first, radiation dominated era, only if their parameter w satisfies $w \geq 1/3$. Thus, their pressure must be comparable with their energy density, and if we want that they are significantly distinct from fluid, their shear modulus must be comparable with their energy density, too.

What kind of matter could have desired properties? One possibility is Coulomb crystal, provided its mobile charges have high enough energies. Coulomb crystals composed of ions with one missing electron and free electrons, both with particle number density n , are known to have shear modulus of order $e^2 n^{4/3}$ [18]. The theory applies also to the highly compressed matter inside white dwarfs and neutron stars crusts, after it has sufficiently cooled down [19, 20]. In this case the standing charges are nuclei rather than ions, therefore the previous expression modifies to $Z^2 e^2 n_0^{4/3}$, where n_0 is the density of nuclei and Z is their atomic number. Consider a crystal consisting of two kinds of *nonelectrically* charged particles, movable ones with the charge $-g$ and standing ones with the charge Zg . (Electrically charged particles, when put into a hot universe, would not form a crystal but a gas of particle-antiparticle pairs with the same temperature as that of radiation.) If $Z \gg 1$, the movable particles can have Fermi energy comparable with the rest mass of the later particles, or even much greater than it, and the standing particles can be still separated by distances greater than their Compton wavelength. Such matter would have energy density of order $n^{4/3}$, where $n = Zn_0$ is the density of the movable particles. The parameter w of the matter would be $1/3$; and the shear modulus would differ from the energy density by the factor $Z^{2/3} g^2$, which might well be of order 1 or greater.

A question arises whether a compressed system of *equally* charged particles cannot be a solid, too. Such system with short-range forces between the particles was considered as a candidate for the matter in the center of neutron stars [21, 22]. If the particle density is large enough, energy density of the system reduces to its potential energy per unit volume and is of order $g^2 m^{-2} n^2$, where g is the charge of the particles and m is the mass of the Yukawa field mediating the interaction between the particles. This corresponds to $w = 1$ (“extremely stiff equation of state”). It could seem that if the particles are arranged into a lattice, after being shifted, they would return to their original positions due to the repulsive forces from their neighbors. If this was the case, the system would be solid rather than fluid. However, it can be shown by direct calculation that

the shear modulus of the lattice is zero. This can be understood in terms of the stress tensor of the Yukawa field: with the $1/r$ singularities removed from the field (since the corresponding stress is compensated by the forces holding the charges together), the tensor is obviously isotropic.

One can also speculate about another kind of solid with positive pressure, a lattice of strings or walls whose energy is *inversely* proportional to their length (in case of strings) or area (in case of walls). Such objects can be called “springlike”, since they behave as a compressed spring. Denote the shear modulus to energy density ratio of the solid by ξ . Lattices of ordinary strings and walls, both random and ordered, were shown to have ξ positive and large enough to make the longitudinal sound speed squared positive [23, 24]. (The cubic lattice considered in [24] has *two* ξ ’s, longitudinal and transversal, defining the corresponding sound speeds. For such lattice, the claim is that both ξ ’s are positive and the longitudinal one yields positive sound speed squared.) Springlike strings and walls, on the other hand, have *negative* ξ ; in particular, when arranged into a random lattice they have $\xi = -2/15$. However, their parameter w is positive, $w = 1/3$ for strings and $w = 2/3$ for walls, and this suffices for the longitudinal sound speed squared to be positive.

Negative ξ seems to be excluded since the transversal sound waves in a solid with such ξ are unstable (their velocity squared is negative). Surprisingly, this might not be as devastating for a solid in the early universe as it would be for a solid in laboratory. Vector perturbations (a different name for transversal sound waves) are usually not considered in cosmology because they fall down as (scale parameter)⁻² [25]. However, in the theories in which vector fields are conformally coupled they are absent completely since they cannot be formed during inflation [26]. Thus, if we restrict ourselves to such theories, instability of vector perturbations does not matter. There is no exponential growth if the initial value of the function is zero.

After the radiation passes a part of its energy to the solid, further evolution of the perturbations depends on how the lattice which was formed in the process looks like. A natural assumption is that it is *relaxed*; that is, it looks locally the same as when created in laboratory and stretched (for $w < 0$) or compressed (for $w > 0$) by the same forces from all sides. It does not follow the shear deformation of the volume elements of the fluid (that is, radiation), since such deformation has no effect on the distribution of the particles as long as they move freely. Instead, the deformation of the fluid translates into the *internal geometry* of the solid (its body geometry in the relaxed state) [1]. Obviously, such isotropic solidification means that no shear stress acts in the solid at the moment it is formed. This lasts until the perturbation starts to evolve, since there is no shear stress without shear deformation. As a result, in a solid with $w > 0$ the presence of ξ shows up only after the perturbation enters the horizon. Long-wavelength perturbations are not affected at all. There is only one way to make these perturbations sensitive to ξ : anisotropic solidification. However, such process cannot take place in common fluid, whose solidification means arranging freely moving particles into a lattice. Anisotropy has to be latently present, inherited from the

process in which the perturbations were created. A scenario to provide a mechanism for that which seems to be most promising is solid inflation proposed in [14, 15, 16, 17].

In the paper we study scalar perturbations in a universe containing solid matter with positive pressure. In section 2 we solve equations governing the evolution of perturbations for a solid with constant parameters w and ξ and match the solutions for a fluid and a solid with the same w ; in section 3 we explore long-wavelength perturbations in a universe containing radiation-like solid ($w = 1/3$); in section 4 we investigate the same kind of perturbations in a universe filled with radiation in which there appears stiff solid ($w > 1/3$); and in section 5 we summarize the results. Signature of the metric tensor is $(+ - - -)$ and a system of units is used in which $c = 16\pi G = 1$.

2 Cosmology with shear stress

2.1 Description of perturbations

Consider a flat FRWL universe filled with an elastic medium, fluid or solid, and add a small perturbation to it. The perturbed metric is

$$ds^2 = a^2(d\eta^2 - d\mathbf{x}^2 + h_{\mu\nu}dx^\mu dx^\nu), \quad (1)$$

where η is conformal time and a is scale parameter. In an unperturbed universe, the medium is characterized by the mass density ρ and pressure p . In a perturbed universe, these quantities acquire small corrections $\delta\rho$ and δp and the medium itself acquires small velocity $\mathbf{v} \doteq a\mathbf{u}$, where \mathbf{u} is the 3-space part of the 4-velocity $u^\mu = dx^\mu/d\tau$. For an ideal fluid, the perturbed stress-energy tensor can be expressed in terms of $\delta\rho$, δp and \mathbf{u} as

$$T_0^0 = \rho + \delta\rho, \quad T_i^0 = a^{-1}\rho_+ u_i, \quad T_i^j = -(p + \delta p)\delta_{ij}, \quad (2)$$

where $\rho_+ = \rho + p$ and $u_i = -a^2 u^i + ah_{0i}$. For a solid, an additional term ΔT_i^j coming from shear stress appears in T_i^j .

Expressions for $\delta\rho$, δp and ΔT_i^j can be obtained from the theory of relativity (relativistic elasticity), summarized in appendix A. Suppose the solid is homogenous and isotropic, so that its elastic properties are completely described by the Lamé coefficients λ and μ . The combination of the two coefficients $K = \lambda + 2\mu/3$ is *compressional modulus* and the coefficient μ is *shear modulus*. (Our K is 2 times greater and our μ is 4 times greater than K and μ in [1]. We have defined them so in order to be consistent with the standard definitions in Newtonian elasticity.) Suppose, furthermore, that the solid has Euclidean internal geometry and no entropy perturbations are present in it, so that the perturbation to the stress-energy tensor is given exclusively by the perturbation to the 3-metric h_{ij} and by the shift vector ξ . Denote the stress tensor and its

perturbation by τ_{ij} and $\delta\tau_{ij}$, $\tau_{ij} = -p\delta_{ij}$ and $\delta\tau_{ij} = -\delta p\delta_{ij} + \Delta T_i^j$. In the comoving gauge $\xi = 0$ and the functions $\delta\rho$ and $\delta\tau_{ij}$ are of the form

$$\delta\rho = \frac{1}{2}\rho_+ h_{kk}, \quad \delta\tau_{ij} = -\frac{1}{2}\lambda h_{kk}\delta_{ij} - \mu h_{ij}. \quad (3)$$

The velocity \mathbf{v} can be written in terms of the shift vector as $\mathbf{v} = \xi'$, where the prime denotes differentiation with respect to η ; thus, in the comoving gauge \mathbf{v} as well as \mathbf{u} vanish and $u_i = ah_{0i}$.

Evolution of an unperturbed universe is described by the equations

$$a' = \left(\frac{1}{6}\rho a^4\right)^{1/2}, \quad \rho' = -3\mathcal{H}\rho_+, \quad (4)$$

where \mathcal{H} is *Hubble parameter* with respect to conformal time, $\mathcal{H} = a'/a$. Note that the equation for ρ follows from the first equation in (A-5) if we insert $V \propto a^3$ into it. From the second equation in (A-5) we obtain in the same way

$$p' = -3\mathcal{H}K. \quad (5)$$

We will be interested in *scalar perturbations* only. These perturbations are most easily interpreted in the *Newtonian gauge*, in which the scalar part of the metric is

$$ds^{(S)2} = a^2[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)d\mathbf{x}^2].$$

Thus, in the Newtonian gauge scalar perturbations to the metric are described solely by two functions, Newtonian potential Φ and an additional potential describing the curvature of 3-space Ψ . Write the scalar part of the 3-tensor $\delta\tau_{ij}$ as

$$\delta\tau_{ij}^{(S)} = \tau^{(1)}\delta_{ij} + \tau^{(2)}_{,ij}.$$

Einstein equations yield three equations for five functions Φ , Ψ , $\delta\rho$, $\tau^{(1)}$ and $\tau^{(2)}$ [25]. We will use just one of them, the algebraical constraint

$$\Phi = \Psi + \frac{1}{2}\tau^{(2)}a^2. \quad (6)$$

Note that an ideal fluid has $\tau^{(2)} = 0$, hence in a universe filled with ideal fluid the potentials Φ and Ψ coincide.

To find the time dependence of Φ and Ψ , one can determine $h_{\mu\nu}$ as functions of η in *any* gauge and compute Φ and Ψ from them. Moreover, one can define the functions $\overline{\delta\rho}$, $\overline{\tau^{(1)}}$ and $\overline{\tau^{(2)}}$ as invariant functions reducing to $\delta\rho$, $\tau^{(1)}$ and $\tau^{(2)}$ in Newtonian gauge, and compute them from the functions $\delta\rho$, $\tau^{(1)}$ and $\tau^{(2)}$ in the gauge one is working in. If we write the scalar part of the perturbed metric in a general gauge as

$$ds^{(S)2} = a^2[(1 + 2\phi)d\eta^2 + 2B_{,i}d\eta dx^i - (\delta_{ij} - 2\psi\delta_{ij} - 2E_{,ij})dx^i dx^j],$$

the expressions for gauge invariant functions are

$$\Phi = \phi - \Delta' - \mathcal{H}\Delta, \quad \Psi = \psi + \mathcal{H}\Delta, \quad \overline{\delta\rho} = \delta\rho - \rho'\Delta, \quad \overline{\tau^{(1)}} = \tau^{(1)} + p'\Delta, \quad (7)$$

where $\Delta = B - E'$ [25]. The function $\tau^{(2)}$ is already gauge invariant, $\overline{\tau^{(2)}} = \tau^{(2)}$.

In our description of perturbations in the presence of solid we will follow [27] and use the *proper-time comoving gauge*, defined by the conditions $\phi = 0$ and $\xi = 0$. In this gauge, the cosmological time $t = \int a d\eta$ is the proper time of the observers at rest and the observers move with the matter. The gauge is not defined uniquely since one can shift cosmological time by an arbitrary function $\delta t(\mathbf{x})$. Under such shift, E stays unaltered and B and ψ transform as

$$B \rightarrow B + \delta\eta, \quad \psi \rightarrow \psi - \mathcal{H}\delta\eta,$$

where $\delta\eta = a^{-1}\delta t$. This suggests that we represent B and ψ as

$$B = \mathcal{B} + \chi, \quad \psi = -\mathcal{H}\chi,$$

where \mathcal{B} stays unaltered by the time shift and χ transforms as $\chi \rightarrow \chi + \delta\eta$. As a result, we obtain an expression for Ψ that is explicitly time-shift invariant,

$$\Psi = \mathcal{H}(\mathcal{B} - E'). \quad (8)$$

For Φ expression (6) will be used, but one can easily check that the expression in (7) is time-shift invariant, too. Indeed, it contains χ only in the combination $a^{-1}(a\chi)'$, and the product $a\chi$ gets shifted by δt which does not depend on η .

Formulas for $\delta\rho$, $\tau^{(1)}$ and $\tau^{(2)}$, obtained by expressing the scalar part of h_{ij} in (3) in terms of ψ and E , are

$$\delta\rho = \rho_+(3\psi + \mathcal{E}), \quad \tau^{(1)} = -3K\psi - \lambda\mathcal{E}, \quad \tau^{(2)} = -2\mu E,$$

where $\mathcal{E} = \Delta E$. After inserting from the first two formulas into the definitions of $\overline{\delta\rho}$ and $\overline{\tau^{(1)}}$ and using expressions (4) and (5) for ρ' and p' , we find

$$\overline{\delta\rho} = \rho_+(3\Psi + \mathcal{E}), \quad \overline{\tau^{(1)}} = -3K\Psi - \lambda\mathcal{E}. \quad (9)$$

Finally, equation (6) with $\tau^{(2)}$ inserted from the third formula reads

$$\Phi = \Psi - \mu a^2 E. \quad (10)$$

We will restrict ourselves to perturbations of the form of plane waves with the wave vector \mathbf{k} , \mathcal{B} and $E \propto e^{i\mathbf{k}\cdot\mathbf{x}}$. The action of the Laplacian then reduces to the multiplication by $-k^2$; in particular, the definition of \mathcal{E} becomes $\mathcal{E} = -k^2 E$. To simplify formulas, we will suppress the factor $e^{i\mathbf{k}\cdot\mathbf{x}}$ in \mathcal{B} and E , as well as in the other functions describing the perturbation. They will be regarded as functions of η only.

For the functions \mathcal{B} and \mathcal{E} we have two coupled linear differential equations of first order, coming from equations $T_i^\mu{}_{;\mu} = 0$ and $2G_{00} = T_{00}$. They can be obtained from equations for y_{01} and y_{11} in [27] by putting $e^z = a$, $y_{01} = a\mathcal{B}$, $y_{11} = -2\mathcal{E}$, $\epsilon = a^3\rho$ and $\sigma = a^3p$ and replacing $\lambda + \sigma \rightarrow a^3\lambda$ and $\mu + \sigma \rightarrow a^3\mu$. The equations are

$$\mathcal{B}' = (3c_{S0}^2 + \alpha - 1)\mathcal{H}\mathcal{B} + c_{S\parallel}^2\mathcal{E}, \quad \mathcal{E}' = -(k^2 + 3\alpha\mathcal{H}^2)\mathcal{B} - \alpha\mathcal{H}\mathcal{E}, \quad (11)$$

where $\alpha = (2\mathcal{H})^{-2}\rho_+a^2$ and the sound speeds c_{S0} and $c_{S\parallel}$ are defined in equations (A-6) and (A-12). Pressure and shear stress of the medium can be characterized by the dimensionless parameters $w = p/\rho$ and $\xi = \mu/\rho$; however, to simplify formulas we will use $\beta = \mu/\rho_+$ instead of ξ . The only place where the parameter β enters equations (11) is the term $c_{S\parallel}^2\mathcal{E}$ in the equation for \mathcal{B} , since $c_{S\parallel}^2 = c_{S0}^2 + (4/3)\beta/$ and c_{S0}^2 does not contain β .

In the concordance model, one does not consider shear stress but introduces another source of anisotropic stress – *viscosity*. Let us compare the effect of viscosity with the effect of shear stress in our gauge. If we introduce one more dimensionless parameter $\gamma = \eta/\rho_+$, where η is the coefficient of shear viscosity, the equation for \mathcal{B} is

$$\mathcal{B}' = (3c_{S0}^2 + \alpha - 1)\mathcal{H}\mathcal{B} + c_{S0}^2\mathcal{E} + \begin{cases} (4/3)\beta\mathcal{E} & \text{for a solid} \\ (4/3)\gamma\mathcal{E}' & \text{for a viscous fluid} \end{cases}.$$

For Φ and Ψ we have equations (8) and (10). After inserting into the former equation from the second equation (11) and into the latter equation from the former, we obtain

$$\Phi_A \equiv (\Phi, \Psi) = -k^{-2}\alpha\mathcal{H}^2(3\mathcal{H}\mathcal{B} + \beta_A\mathcal{E}), \quad \beta_A = (1 - 4\beta, 1). \quad (12)$$

2.2 Solution for a one-component medium

Consider a universe filled with a one-component elastic medium whose characteristics p , λ and μ are all proportional to ρ . From equations (4) we find

$$\rho \propto a^{-3w_+}, \quad a \propto \eta^{2u},$$

where $w_+ = 1 + w$ and $u = 1/(1 + 3w)$. (We suppose that $w > -1/3$, otherwise we should write $a \propto (\text{sign } \eta)^{2u}$.) Note that since the compressional modulus can be written as $K = dp/d\rho \rho_+$, it *must be* proportional to ρ , $K = w\rho_+ = ww_+\rho$. Thus, we must require, besides that w is constant, only that β is constant. If this is the case, the functions appearing in (11) are all constant, except for the function \mathcal{H} which is proportional to η^{-1} ,

$$\alpha = \frac{3}{2}w_+, \quad c_{S0}^2 = w, \quad c_{S\parallel}^2 = w + \frac{4}{3}\beta \equiv \tilde{w}, \quad \mathcal{H} = 2u\eta^{-1}.$$

After expressions for α , c_{S0}^2 , $c_{S\parallel}^2$ and \mathcal{H} are inserted into equations (11), they transform into

$$\mathcal{B}' = u(1 + 9w)\eta^{-1}\mathcal{B} + \tilde{w}\mathcal{E}, \quad \mathcal{E}' = -(k^2 + 18u^2w_+\eta^{-2})\mathcal{B} - 3uw_+\eta^{-1}\mathcal{E}. \quad (13)$$

The two equations of first order for \mathcal{B} and \mathcal{E} can be combined into one equation of second order for \mathcal{B} ,

$$\mathcal{B}'' + 2\nu_B \eta^{-1} \mathcal{B}' + [q^2 - (2\nu_B - b)\eta^{-2}] \mathcal{B} = 0, \quad (14)$$

where $q = \sqrt{\tilde{w}}k$, $\nu_B = u(1 - 3w)$ and $b = 24u^2 w_+ \beta$. Furthermore, \mathcal{E} can be expressed in terms of \mathcal{B} and \mathcal{B}' , and by using (12), Φ and Ψ can be expressed in terms of \mathcal{B} and \mathcal{B}' , too. We obtain

$$\Phi_A = -\beta_A \Sigma (q\eta)^{-2} [\mathcal{B}' - (1 - \sigma_A)\eta^{-1} \mathcal{B}], \quad (15)$$

where $\Sigma = 6u^2 w_+$ and $\sigma_A = ((1 - 4\beta)^{-1}, u) 8\beta$.

Solution to equation (14) is

$$\mathcal{B} = z^{-\nu_-} (c_J J_n + c_Y Y_n), \quad (16)$$

where $z = q\eta$, ν_- and n are defined in terms of $\nu = \nu_B + 1/2 = 3u(1 - w)/2$ as $\nu_- = \nu - 1$ and $n = \sqrt{\nu^2 - b}$, and J_n and Y_n are Bessel functions of first and second kind of the argument z . Note that since $q = c_{S\parallel} k$, the value $z = 1$ corresponds to the moment at which the perturbation crosses the sound horizon (its reduced wavelength becomes less than the radius of the horizon).

Knowing the function \mathcal{B} we can determine the potentials Φ and Ψ . Denote $\nu_+ = \nu + 1$, $n_+ = n + 1$ and $m = \nu - n$. By inserting (16) into (15) and using the identities

$$\frac{dJ_n}{dz} = -J_{n_+} + n z^{-1} J_n, \quad \frac{dY_n}{dz} = -Y_{n_+} + n z^{-1} Y_n,$$

we obtain

$$\Phi_A = \beta_A z^{-\nu_+} \{C_J [J_{n_+} + (m - \sigma_A) z^{-1} J_n] + C_Y [Y_{n_+} + (m - \sigma_A) z^{-1} Y_n]\}, \quad (17)$$

where C_J and C_Y are defined in terms of c_J and c_Y as $C_J = \Sigma q c_J$ and $C_Y = \Sigma q c_Y$. Another important quantity is *density contrast* $\delta = \overline{\delta\rho}/\rho$. To determine it, we need to know the function \mathcal{E} . By using the identities for dJ_n/dz and dY_n/dz once again we find

$$\mathcal{E} = z^{-\nu_-} \{\hat{c}_J [J_{n_+} + (m + \tau) z^{-1} J_n] + \hat{c}_Y [Y_{n_+} + (m + \tau) z^{-1} Y_n]\}, \quad (18)$$

where $\tau = 6uw$ and \hat{c}_J and \hat{c}_Y are defined in terms of c_J and c_Y as $\hat{c}_J = -\tilde{w}^{-1} q c_J$ and $\hat{c}_Y = -\tilde{w}^{-1} q c_Y$. The density contrast is obtained by inserting this expression along with the expression for $\Psi \equiv \Phi_2$ into

$$\delta = w_+ (3\Psi + \mathcal{E}). \quad (19)$$

An ideal fluid has $\beta = 0$, hence $n = \nu$, $m = \sigma_A = 0$ and

$$\Phi = \Psi = z^{-\nu_+} (C_J J_{\nu_+} + C_Y Y_{\nu_+}). \quad (20)$$

This agrees with the formula (7.58) in [25] if we realize that $z = \sqrt{\tilde{w}}k\eta$ for $\beta = 0$ and $\nu_+ = u(5 + 3w)/2$ for any β .

Let us determine the asymptotics of the functions \mathcal{B} , Φ , Ψ and \mathcal{E} at $z \ll 1$ (in the first period after the perturbation was formed, when its wavelength exceeded the size of the horizon considerably). Denote the coefficients in the leading terms in J_n and Y_n by J and Y ,

$$J = \frac{1}{2^n \Gamma(n_+)}, \quad Y = -\frac{1}{\pi} 2^n \Gamma(n),$$

and introduce one more parameter $M = \nu + n$. With these notations we have

$$\mathcal{B} \doteq z(c_J J z^{-m} + c_Y Y z^{-M}). \quad (21)$$

Introduce, furthermore, the coefficients in the leading terms in J_{n_+} and Y_{n_+} , $J_+ = J/(2n_+)$ and $Y_+ = 2nY$. Then it holds

$$\Phi_A \doteq \beta_A [C_J J_+ (P_A z^{-m} + Q_A z^{-2-m}) + C_Y Y_+ R_A z^{-2-M}], \quad (22)$$

where

$$P_A = 1 - \frac{1}{2}(m - \sigma_A), \quad Q_A = 2n_+(m - \sigma_A), \quad R_A = 1 + \frac{1}{2n}(m - \sigma_A),$$

and

$$\mathcal{E} \doteq \hat{c}_J J_+ Q z^{-m} + \hat{c}_Y Y_+ R z^{-M}, \quad (23)$$

where

$$Q = 2n_+(m + \tau), \quad R = 1 + \frac{1}{2n}(m + \tau).$$

We have included the P -term into Φ_A although it is of higher order in z than the Q -term. The reason is that for $|\beta| \ll 1$, P_A as well as $R_A \doteq 1$ while $Q_A = O(\beta)$. Thus, if $|\beta|$ is small, the P -term can prevail over the Q -term starting at some value of z that is still small.

For the sake of completeness, note that the estimate of Q_A refers to the case when w is not close to zero. If it is, the estimate is lower: for $|\beta| \ll |w| \ll 1$ it holds $Q_A = O(w\beta)$ and for $|w| \ll |\beta|$ it holds $Q_A = O(\beta^2)$. This does not affect the above argument, since it is using just the fact that Q_A is small for $|\beta|$ small.

For an ideal fluid the asymptotics reduce to

$$\mathcal{B} \doteq z(c_J J + c_Y Y z^{-2\nu}), \quad (24)$$

$$\Phi = \Psi \doteq C_J J_+ + C_Y Y_+ z^{-2\nu_+}, \quad (25)$$

and

$$\mathcal{E} \doteq \hat{c}_J J_+ Q_{id} + \hat{c}_Y Y_+ R_{id} z^{-2\nu}, \quad (26)$$

where Q_{id} and R_{id} are the values of Q and R for $\beta = 0$, $Q_{id} = 2\nu_+ \tau$ and $R_{id} = 1 + \tau/(2\nu)$.

2.3 Switching the shear stress at a finite time

Suppose the universe was originally filled with an ideal fluid with the given value of w and then, at some moment η_s , all fluid instantaneously turned into a solid with the same w . Let the transition be anisotropic, so that the solid was formed with *Euclidean* internal geometry. Perturbations in such universe are described by equations (11), in which the parameter β must be replaced by the function $\beta\theta(\eta - \eta_s)$. The equations imply that the functions \mathcal{B} and \mathcal{E} are both continuous at η_s and the derivative of \mathcal{E} is continuous, too, while the derivative of \mathcal{B} has a jump coming from the jump in $c_{S\parallel}^2$. From equations (8) and (10) we can also see that the function Ψ is continuous with a jump in its derivative, while the function Φ has a jump itself.

The function \mathcal{B} is given by equation (16) both in the ideal fluid and solid state era. However, in the latter era the constants c_J and c_Y , and even the variable z , are different than in the former era. The constants change in order to satisfy matching conditions and the variable changes because it contains the parameter q that switches from the value $q_0 = \sqrt{w}k$ to the value $q = \sqrt{\bar{w}}k$. Thus, if we write the function \mathcal{B} in the solid state era as in (16), we must write it in the ideal fluid era as

$$\mathcal{B}_0 = z_0^{-\nu-} (c_{J0} J_\nu + c_{Y0} Y_\nu),$$

where $z_0 = q_0\eta$ and J_ν and Y_ν are Bessel functions of first and second kind of the argument z_0 . The function \mathcal{E}_0 , needed for the matching procedure, is obtained in the same way from the ideal fluid version of (18).

The matching conditions read

$$\mathcal{B}_s = \mathcal{B}_{0s}, \quad \mathcal{B}'_s = \mathcal{B}'_{0s} + \frac{4}{3}\beta\mathcal{E}_{0s}, \quad (27)$$

where the index s indicates that the function is evaluated at the moment η_s . The first condition states that \mathcal{B} is continuous at the moment η_s and the second condition fixes the jump in the derivative of \mathcal{B} in accordance with the first equation in (11).

Suppose $z_{0s} = q_0\eta_s \ll 1$; thus, the perturbation is by assumption stretched far beyond the sound horizon at the moment the shear stress switches on. Suppose, furthermore, that $z_s = q\eta_s \ll 1$; thus, the perturbation stays stretched beyond the sound horizon also during some period after the shear stress switched on. (This is nontrivial in case $|w| \ll 1$ and $\beta \sim 1$, since then $q_0 \ll q$.) The assumptions simplify the form of the functions entering the matching conditions considerably. All three are given by the asymptotic formulas valid at $z \ll 1$, the functions \mathcal{B}_0 and \mathcal{E}_0 by equations (24) and (26) and the function \mathcal{B} by equation (21).

The ratio of the second to the first term in the asymptotic formulas for \mathcal{B}_0 and \mathcal{E}_0 varies with z_0 as $z_0^{-\nu}$. Let w be from the interval $(-1/3, 1)$. The parameter ν is then positive and the ratio decays with time. Let us simplify the theory even more by assuming that both terms were about the same at the moment when the perturbation was formed, and that the shear stress was switched

long enough after that moment. Then we are left with \mathcal{B}_0 and \mathcal{E}_0 containing the nondecaying term only,

$$\mathcal{B}_0 = c_{J0}J_0z_0 \equiv C_0z_0, \quad \mathcal{E}_0 = \hat{c}_{J0}J_{0+}Q_{id,0} = -6uq_0C_0. \quad (28)$$

After inserting these expressions into the combination of \mathcal{B}'_0 and \mathcal{E}_0 that appears on the right hand side of the second matching condition, we find

$$\mathcal{B}'_0 + \frac{4}{3}\beta\mathcal{E}_0 = q_0C_0(1 - \sigma_2).$$

On the left hand side of the matching conditions we retain both terms appearing in the asymptotic formula for \mathcal{B} , the term proportional to z^{1-m} as well as the term proportional to z^{1-M} . As a result, we obtain

$$x + y = C, \quad (1 - m)x + (1 - M)y = C(1 - \sigma_2), \quad (29)$$

where $C = q_0C_0/q$ and x and y are defined in terms of c_J and c_Y as $x = c_JJz_s^{-m}$ and $y = c_Y Yz_s^{-M}$. The solution is

$$x = \frac{C}{2n}(M - \sigma_2), \quad y = -\frac{C}{2n}(m - \sigma_2). \quad (30)$$

Instead of the constant C , it is more convenient to use Newtonian potential of the perturbation in the ideal fluid era Φ_0 . If we evaluate Φ_0 under the same assumptions as \mathcal{B}_0 and \mathcal{E}_0 , we obtain $\Phi_0 = C_{J0}J_{0+}$, and since $C_{J0} = \Sigma q_0 c_{J0} = \Sigma q_0 C_0/J_0 = \Sigma q C/J_0$ and $J_{0+} = J_0/(2\nu_+)$, we have

$$\Phi_0 = \frac{\Sigma q}{2\nu_+}C. \quad (31)$$

Using this relation together with the identities

$$M - \sigma_2 = 2nR_2, \quad m - \sigma_2 = \frac{1}{2n_+}Q_2,$$

we obtain

$$x = \frac{2\nu_+}{\Sigma q}R_2\Phi_0, \quad y = -\frac{\nu_+}{2nn_+\Sigma q}Q_2\Phi_0. \quad (32)$$

For further reference, let us also express the density contrast in the ideal fluid era δ_0 in terms of Φ_0 . It holds

$$\mathcal{E}_0 = -\frac{12u\nu_+}{\Sigma}\Phi_0 = -\frac{5+3w}{w_+}\Phi_0,$$

and by inserting this into the formula $\delta_0 = w_+(3\Phi_0 + \mathcal{E}_0)$ we obtain, irrespective of the value of w ,

$$\delta_0 = -2\Phi_0. \quad (33)$$

The asymptotics of Φ , Ψ and \mathcal{E} can be found from equations (22) and (23) by computing c_JJ and $c_Y Y$ from x and y and using the formulas $C_JJ_+ = \Sigma qc_JJ/(2n_+)$, $C_Y Y_+ = 2n\Sigma qc_Y Y$, and $\hat{c}_J = -(\tilde{w}\Sigma)^{-1}C_J$, $\hat{c}_Y = -(\tilde{w}\Sigma)^{-1}C_Y$. If we also introduce the rescaled time $\zeta = \eta/\eta_s = z/z_s$, we obtain

$$\Phi_A \doteq \mathcal{P}_A\zeta^{-m} + \mathcal{Q}_Az_s^{-2}\zeta^{-2-m} + \mathcal{R}_Az_s^{-2}\zeta^{-2-M}, \quad (34)$$

where

$$\mathcal{P}_A = \beta_A \frac{\nu_+}{n_+} P_A R_2 \Phi_0, \quad \mathcal{Q}_A = \beta_A \frac{\nu_+}{n_+} Q_A R_2 \Phi_0, \quad \mathcal{R}_A = -\beta_A \frac{\nu_+}{n_+} Q_2 R_A \Phi_0,$$

and

$$\mathcal{E} \doteq -\mathcal{Q}\zeta^{-m} - \mathcal{R}\zeta^{-M}, \quad (35)$$

where

$$\mathcal{Q} = (\tilde{w}\Sigma)^{-1} \frac{\nu_+}{n_+} Q R_2 \Phi_0, \quad \mathcal{R} = -(\tilde{w}\Sigma)^{-1} \frac{\nu_+}{n_+} Q_2 R \Phi_0.$$

Note that the coefficients \mathcal{Q}_2 and \mathcal{R}_2 in the expression for $\Psi \equiv \Phi_2$ are the same except for their sign. This guarantees that Ψ does not have jump of order $z_s^{-2}\Phi_0$ at $\eta = \eta_s$.

In the formula for Φ_A , the coefficients \mathcal{Q}_A and \mathcal{R}_A are multiplied by z_s^{-2} . Thus, at the moment η_s the P -term is suppressed with respect to the Q - and R -terms by the factor z_s^2 . On the other hand, if $|\beta|$ is small, the coefficients \mathcal{Q}_A and \mathcal{R}_A are of order β , \mathcal{Q}_A because of the factor Q_A appearing in the original formula and \mathcal{R}_A because of the factor Q_2 coming from the expression for c_J . Thus, at the moment η_s the P -term is enhanced with respect to the Q - and R -terms by the factor β^{-1} . However, we have solved the matching conditions only in the leading order in z_s . The resulting theory is therefore applicable only if the net effect is *suppression* of the P -term at the time η_s ; the term can eventually prevail, but only at times much greater than η_s . This leads to the condition $|\beta| \gg z_s^2$. Since the estimate of Q_A holds only for w not too close to zero, so does the constraint on β . If we take into account the behavior of Q_A for w close to zero, we arrive at a stronger condition $|\beta| \gg \min\{z_s^2/|w|, z_s\}$.

To justify our matching procedure, let us show that it is consistent with the description of scalar perturbations in Newtonian gauge. Equation with $\overline{\tau^{(1)}}$ on the right hand side obtained in that gauge reads (see equation (7.40) in [25])

$$\Psi'' + \mathcal{H}(2\Psi' + \Phi') + (2\mathcal{H}' + \mathcal{H}^2)\Psi - \frac{1}{2}k^2(\Phi - \Psi) = -\frac{1}{4}\overline{\tau^{(1)}}. \quad (36)$$

Denote the jump of the function at the moment η_s by square brackets. From equation (12) with $A = 1$ we find

$$[\Phi] = 4k^{-2}\alpha_s\mathcal{H}_s^2\beta\mathcal{E}_s.$$

Equation (12) with $A = 2$ yields $[\Psi'] = -3k^{-2}\alpha_s\mathcal{H}_s^3[\mathcal{B}']$, and if we use the formula $[\mathcal{B}'] = 4\beta/3\mathcal{E}_s$, following from the first equation in (11), we have

$$[\Psi'] = -\mathcal{H}_s[\Phi]. \quad (37)$$

The terms Ψ'' and $\mathcal{H}\Phi'$ on the left hand side of (36) both contain δ -function; however, the identity we have obtained ensures that the δ -functions cancel and only a jump-like discontinuity remains.

3 Radiation-like solid

3.1 Perturbations in a universe with radiation-like solid

In standard cosmology the universe was dominated by the radiation from the end of inflation almost up to recombination. Radiation is ideal fluid with $w = 1/3$, therefore to make our problem more realistic we must suppose that the value of w in the ideal fluid era was $1/3$. Then we can use the previous theory without modifications, if we require that the value of w in the solid state era was $1/3$, too. This means that a portion of radiation has been eventually converted into a solid with the same pressure to energy density ratio. We will call such solid *radiation-like*.

Evolution of perturbations in the presence of radiation-like solid is given by the formulas derived in the previous section, with $w = 1/3$ inserted everywhere. The constants entering the formulas are $u = \nu = 1/2$, $\Sigma = 2$, $\tau = 1$ and $b = 8\beta$. If we also rewrite β_A , σ_A and \tilde{w} in terms of b and insert the value of ν into the definition of n , we obtain for the remaining constants $\beta_A = (1 - b/2, 1)$, $\sigma_A = ((1 - b/2)^{-1}, 1/2)b$, $\tilde{w} = (1 + b/2)/3$ and $n = \sqrt{1/4 - b}$.

The parameter \tilde{w} must be positive in order that longitudinal sound waves are stable, and the parameter n must be real in order that the evolution of perturbations in a universe filled with pure solid is smooth from the beginning. As a result, b must be from the interval $(-2, 1/4)$. However, as we will see, the theory agrees with observations only if b is close to zero. For such b it holds $n \doteq 1/2 - b$, $m = 1/2 - n \doteq b$ and $M = 1/2 + n \doteq 1 - b$, and the constants in the asymptotic formulas for the functions Φ , Ψ and \mathcal{E} , evaluated in the leading order in b , are

$$(P_1, Q_1, R_1) \doteq \left(1, \frac{3}{2}b^2, 1\right), \quad (P_2, Q_2, R_2) \doteq \left(1, \frac{3}{2}b, 1\right), \quad (Q, R) \doteq (3, 2).$$

While Q_2 is of first order in b as expected, Q_1 turns out to be of *second* order. The value $w = 1/3$ is special in this respect. (The only other value for which Q_1 is of second order is $w = 0$, but Q_2 is for that w of second order, too.) When arguing that the constraint $|b| \gg z_s^2$ must be observed in order that the expressions for (P_A, Q_A, R_A) are valid, we have assumed that both Q_A are of first order in b . The fact that Q_1 is of second order does not lead to strengthening of this constraint; expressions for (P_1, Q_1, R_1) can be used also for $|b| \lesssim z_s$ because the P -term in Φ , while not suppressed with respect to the Q -term at the moment η_s , is still suppressed with respect to the R -term. After inserting these expressions into the formulas for $(\mathcal{P}_A, \mathcal{Q}_A, \mathcal{R}_A)$ and $(\mathcal{Q}, \mathcal{R})$ and using the approximate equalities $\beta_A \nu_+ / n_+ \doteq (1, 1)$ and $(\tilde{w}\Sigma)^{-1} \nu_+ / n_+ \doteq 3/2$, we obtain

$$(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1) \doteq \left(1, \frac{3}{2}b^2, -\frac{3}{2}b\right)\Phi_0, \quad (\mathcal{P}_2, \mathcal{Q}_2, \mathcal{R}_2) \doteq \left(1, \frac{3}{2}b, -\frac{3}{2}b\right)\Phi_0, \quad (\mathcal{Q}, \mathcal{R}) \doteq \frac{9}{2}(1, -b)\Phi_0.$$

As a result, approximate expressions for the functions Φ , Ψ and \mathcal{E} in the regime in which the perturbation is stretched far beyond the sound horizon are

$$\Phi \doteq \left[\zeta^{-b} + \frac{3}{2}bz_s^{-2}(b\zeta^{-2-b} - \zeta^{-3+b})\right]\Phi_0, \quad \Psi \doteq \left[\zeta^{-b} + \frac{3}{2}bz_s^{-2}(\zeta^{-2-b} - \zeta^{-3+b})\right]\Phi_0, \quad (38)$$

and

$$\mathcal{E} \doteq -\frac{9}{2}(\zeta^{-b} - b\zeta^{-1+b})\Phi_0. \quad (39)$$

Knowing the functions Ψ and \mathcal{E} , we can compute the density contrast as $\delta = 4(\Psi + \mathcal{E}/3)$.

The dependence of the functions $\tilde{\Phi} = \Phi/\Phi_0$, $\tilde{\Psi} = \Psi/\Phi_0$ and $\tilde{\delta} = \delta/\Phi_0$ on time is shown in fig. 1. The rescaled density contrast is multiplied by $-1/2$ in order to normalize it to 1 in the ideal fluid era (see equation (33)). The value of the dimensionless shear modulus is $b = 0.01$ in the left

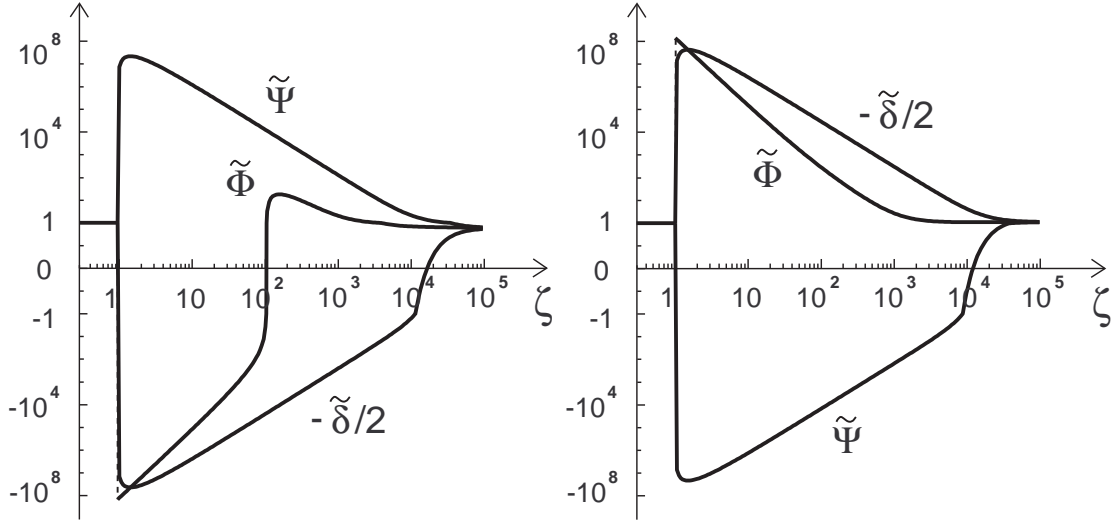


Fig. 1: Behavior of gravitational potentials and density contrast for positive (left) and negative (right) shear stress

panel and $b = -0.01$ in the right panel, and the value of the variable z at the moment when the shear stress is switched on is $z_s = 10^{-5}$ in both panels. The scale is linear in the central band and logarithmic outside of it. The variable ζ is bounded from above by the value $z_s^{-1} = 10^5$, at which $z = 1$ and the perturbation crosses the sound horizon. Up to that point, we have evaluated the functions $\tilde{\Phi}$, $\tilde{\Psi}$ and $\tilde{\mathcal{E}}$ from the asymptotic formulas (38) and (39) with suppressed factor Φ_0 . This is an extrapolation, since the formulas are applicable only at $z \ll 1$. At $z \sim 1$ the functions begin to oscillate; thus, the curves computed from exact formulas bend down at the right edge of the panels instead of being approximately horizontal.

The function $\tilde{\Phi}$ jumps from 1 to $-3b/2 z_s^{-2} = \mp 1.5 \times 10^8$ at $\zeta = 1$. (All expressions in this paragraph are approximate, with the leading term cited only.) The function $\tilde{\Psi}$ is continuous, however, it rises abruptly with rising ζ in case $b > 0$ and falls down abruptly with rising ζ in case $b < 0$, so that it assumes a comparable value with opposite sign at a nearby ζ . Its maximum/minimum is reached at $\zeta = 3/2$ and equals $2b/9 z_s^{-2} = \pm 2.2 \times 10^7$. As ζ increases, $\tilde{\Psi}$ decreases as ζ^{-2-b} in case $b > 0$ and increases as $-\zeta^{-2-b}$ in case $b < 0$, and then, after ζ reaches

the value $|b|^{1/2}z_s^{-1} = 10^4$, it relaxes to the function ζ^{-b} , which is approximately constant and equal to 1 in the interval of ζ under consideration. The function $-\tilde{\delta}/2$ coincides approximately with $-\tilde{\Psi}$ as long as ζ stays less than $|b|^{1/2}z_s^{-1}$. Thus, it reaches the value $\mp 4.4 \times 10^7$ at $\zeta = 3/2$ and then it increases as $-\zeta^{-2-b}$ in case $b > 0$ and decreases as ζ^{-2-b} in case $b < 0$. After ζ rises above $|b|^{1/2}z_s^{-1}$, it relaxes to the function ζ^{-b} just as $\tilde{\Psi}$ does. We can summarize this behavior by saying that $\tilde{\delta}$ switches from $2\tilde{\Psi}$ in the first regime to $-2\tilde{\Psi}$ in the second regime. Finally, the function $\tilde{\Phi}$ operates in three regimes. For $\zeta \lesssim |b|^{-1} = 100$ it increases as $-\zeta^{-3+b}$ in case $b > 0$ and decreases as ζ^{-3+b} in case $b < 0$; for $|b|^{-1} \lesssim \zeta \lesssim |b|z_s^{-1} = 1000$ it decreases as ζ^{-2-b} in both cases; and for $\zeta \gtrsim |b|z_s^{-1}$ it relaxes to ζ^{-b} . In case $b > 0$, it reaches maximum when passing from the first regime to the second one. The maximum occurs at $\zeta = 3/(2b) = 150$ and the value of $\tilde{\Phi}$ is $2b^4/9 z_s^{-2} = 22$.

3.2 Size of perturbations

Perturbations must be small in order that the linearized theory describing them is applicable. However, the size of the perturbations depends on gauge. By changing it, one can surely turn a small perturbation into a large one. Therefore the smallness of perturbations must be defined as an existence property: the perturbation is small if there *exists* gauge in which it is small.

To estimate the size of perturbations we need to know the value of Φ_0 . Consider the perturbed universe at the moment of recombination η_{re} . We can describe the behavior of perturbations up to that moment *approximately* by the asymptotic formulas (38) and (39). The description is approximate since we are ignoring the fact that the universe becomes matter-dominated in the last period before η_{re} . Because of that, the parameter w does not stay constant; it falls down from $1/3$ to approximately $1/12$. Suppose the perturbation crosses the sound horizon at recombination; in other words, suppose the wave number k is such that the moment of horizon crossing $\eta_h = q^{-1} = (\tilde{w}k)^{-1}$ equals η_{re} . As seen from fig. 1, this yields values of Φ_{re} , Ψ_{re} and $-\delta_{re}/2$ close to Φ_0 (because the values of $\tilde{\Phi}$, $\tilde{\Psi}$ and $-\tilde{\delta}/2$ at $\zeta = z_s^{-1}$ are close to 1). On the other hand, from the magnitude of CMB anisotropies we know that the quantities Φ_{re} , Ψ_{re} and δ_{re} are of order 10^{-5} for long-wavelength perturbation. Thus, the value of Φ_0 for the perturbations under consideration must be of order 10^{-5} , too. Inflation yields flat initial spectrum, so that if we accept inflation as the mechanism by which the perturbations were created, Φ_0 must be of order 10^{-5} for *all* perturbations.

The maximum the functions Φ , Ψ and δ reach in absolute value is of order $|b|z_s^{-2}\Phi_0$. (We suppose $\Phi_0 > 0$.) As a result, if we describe the perturbed universe by these functions, the requirement that the perturbation is small leads to the constraint on the dimensionless shear modulus $|b| \ll z_s^2\Phi_0^{-1} \approx 10^5 z_s^2$. To get an idea of how strong the constraint is, consider perturba-

tions crossing the sound horizon at recombination. By definition, the parameter z_s equals $q\eta_s$, or η_s/η_h , hence its value for such perturbations is $z_s^{(0)} = \eta_s/\eta_{re}$. The time dependence of the scale parameter in a universe filled with radiation is $a \propto \eta$, so that $z_s^{(0)} = a_s/a_{re}$; or, if we denote the temperature of cosmic medium by T , $z_s^{(0)} = T_{re}/T_s$. Thus, $z_s^{(0)}$ is the ratio of two energy scales, the scale of recombination and the scale of shear switch. The value $z_s = 10^{-5}$ we have used in our illustrative computation, if identified with $z_s^{(0)}$, corresponds with the scale of shear switch 0.1 MeV. In other words, $z_s^{(0)}$ assumes this value if the radiation-like solid has been formed not earlier than at the time of nucleosynthesis. The resulting constraint on the shear stress is $|b| \ll 10^{-5}$. If the solid appears at an earlier stage, the constraint becomes stronger. In particular, formation of the solid on the GUT scale leads to $z_s^{(0)} = 10^{-23}$ and $|b| \ll 10^{-41}$.

The previous analysis can be easily extended to long-wavelength perturbations. If we denote the wave number of perturbations crossing the sound horizon at recombination by $k^{(0)}$, the perturbations with the longest wavelength that can be observed in CMB have $k \doteq 0.01k^{(0)}$. Thus, their z_s is by two orders of magnitude less than $z_s^{(0)}$, and to keep them small, we must restrict the value of $|b|$ by a number that is by four orders of magnitude smaller than the numbers cited above. We can see that to guarantee that the functions Φ , Ψ and δ are close to zero, the universe must be filled with matter that is practically indistinguishable from an ideal fluid.

The functions Φ , Ψ and δ have been defined in a gauge invariant way, but their invariance is *computational*, not *conceptual*. They can be calculated in any gauge by using the formulas (7), but they refer to a *particular* gauge, namely to Newtonian gauge defined by the conditions $h_{0i} = E = 0$. Thus, the constraints we have established by requiring that Φ , Ψ and δ are small in absolute value are in fact gauge dependent. To find out whether they cannot be relaxed, let us look at the constraints in the proper-time comoving gauge, defined by the conditions $\phi = \xi = 0$.

The scalar part of the perturbation to the metric in the proper-time comoving gauge is given by the three functions kB (since $h_{0i}^{(S)} = B_{,i} = ikB$), ψ and $\mathcal{E} = -k^2E$ (since $h_{ij}^{(S)}$ is the sum of $h_{ij}^{(S1)} = 2\psi\delta_{ij}$ and $h_{ij}^{(S2)} = 2E_{,ij} = -2k_ik_jE$). Perturbation to the matter density equals $4(\psi + \mathcal{E}/3)$ and does not need to be considered separately. For the function \mathcal{E} we have expression (35), which for $w = 1/3$ and $|b| \ll 1$ reduces to (39). The function $k\mathcal{B}$ is given by equation (21), which transforms after inserting for c_J and c_Y into

$$k\mathcal{B} \doteq z(\mathcal{Q}_b\zeta^{-m} + \mathcal{R}_b\zeta^{-M}), \quad (40)$$

where

$$\mathcal{Q}_b = (\sqrt{\tilde{w}\Sigma})^{-1}2\nu_+R_2\Phi_0, \quad \mathcal{R}_b = -(\sqrt{\tilde{w}\Sigma})^{-1}\frac{\nu_+}{2nn_+}Q_2\Phi_0,$$

For $w = 1/3$ and $|b| \ll 1$ this reduces to

$$k\mathcal{B} \doteq \frac{3\sqrt{3}}{2}z\left(\zeta^{-b} - \frac{1}{2}b\zeta^{-1+b}\right)\Phi_0. \quad (41)$$

The function χ appearing in the expression for kB satisfies an equation coming from the longitudinal part of the equation $2G_{0i} = T_{0i}$. It can be obtained from the equation for y in [27] by putting $y = a\chi$, and is of the form

$$\chi' = -\mathcal{H}(\chi + \alpha\mathcal{B}).$$

The solution is

$$\chi = -a^{-1} \int \alpha\mathcal{B} da, \quad (42)$$

and if we insert here $\alpha = 2$ and $a \propto \eta$ and use the approximate expression (41) for $k\mathcal{B}$, we obtain

$$k\chi \doteq -\frac{3\sqrt{3}}{2}z \left[\left(1 + \frac{b}{2}\right)\zeta^{-b} - b\zeta^{-1+b} \right] \Phi_0. \quad (43)$$

We have skipped the term with integration constant since it can be always removed by an appropriate choice of the start of time counting. On the other hand, we have included correction of order b into the term in square brackets proportional to ζ^{-b} , in order that we are able to calculate the leading term proportional to ζ^{-b} in the function $kB = k(\mathcal{B} + \chi)$. Of course, we do not need this correction when computing the function $\psi = -\eta^{-1}\chi = -\sqrt{w}z^{-1}k\chi$. After inserting into the definitions of kB and ψ from equations (41) and (43), we find

$$kB \doteq -\frac{3\sqrt{3}}{4}bz(\zeta^{-b} - \zeta^{-1+b})\Phi_0, \quad \psi \doteq \frac{3}{2}(\zeta^{-b} - b\zeta^{-1+b})\Phi_0. \quad (44)$$

We are interested in the behavior of the functions kB , ψ and \mathcal{E} for ζ ranging from 1 (the time the solid appeared) to $(z_s^{(0)})^{-1}$ (the time of recombination). The formulas (39) and (44) hold, with greater or less accuracy, on the whole interval of ζ if $k \leq k^{(0)}$, and up to $\zeta = z_s^{-1} = (k/k^{(0)})^{-1} \times$ the maximum ζ if $k > k^{(0)}$. Outside that interval all three functions oscillate, kB and \mathcal{E} with constant amplitude and ψ with falling amplitude. (The two terms in $k\mathcal{B}$ and the leading two terms in \mathcal{E} are of the form $z^{1/2} \times$ Bessel function, and Bessel functions oscillate with the amplitude $\sim z^{-1/2}$; thus, $k\mathcal{B}$ and \mathcal{E} oscillate with constant amplitude, χ oscillates with amplitude proportional to z^{-1} and ψ oscillates with amplitude proportional to z^{-2} .) Clearly, if we want to estimate the functions kB , ψ and \mathcal{E} in absolute value from above, for $k > k^{(0)}$ we can use the same formulas as for $k \leq k^{(0)}$, only on a smaller interval. We will restrict ourselves to perturbations with $k = k^{(0)}$; however, it can be easily checked by using the effective domain of kB , ψ and \mathcal{E} that the results stay the same after one extends the analysis to perturbations with arbitrary k .

The functions ζ^{-b} and ζ^{-1+b} both equal 1 at $\zeta = 1$, and since $|b| \ll 1$, the second function always decreases while the first function either decreases at a slower rate or increases. Thus, the first term in kB dominates the second term except for values of ζ comparable with 1, and the first term in ψ and \mathcal{E} dominates the second term for all ζ . The order of magnitude of kB is given by its first term even for ζ of order 1, except for a small interval close to 1 in which kB is close to zero. As a result, kB is in absolute value almost everywhere of order $|b|z\zeta^{-b}\Phi_0$ and ψ and \mathcal{E} are in

absolute value everywhere of order $\zeta^{-b}\Phi_0$. If $b > 0$, the maximum of the function $z\zeta^{-b} = z_s^{(0)}\zeta^{1-b}$ for $k = k^{(0)}$ is $(z_s^{(0)})^b$, which is less than 1, and the maximum of the function ζ^{-b} is 1; if $b < 0$, the maximum of both functions is $(z_s^{(0)})^b$, which is greater than 1. We arrive at the conclusion that for $b > 0$ the requirement of smallness of perturbations does not give any constraint on b , while for $b < 0$ it leads to the constraint $(z_s^{(0)})^b \ll \Phi_0^{-1} \approx 10^5$, or $b \gtrsim 5/\log_{10}(z_s^{(0)})$. If we extrapolate the theory to finite values of b , we find that b must be slightly greater than -1 if the solid appeared at the scale of nucleosynthesis, and slightly greater than -0.22 if it appeared at the GUT scale. These conditions are *substantially* weaker than the ones we have obtained in Newtonian gauge.

The analysis of the behavior of the functions kB , ψ and \mathcal{E} shows that if the functions Φ , Ψ and δ in a universe with solid component become in absolute value comparable with 1, or even much greater than 1, that does not mean that the theory has collapsed. Such behavior means simply that Newtonian gauge is not appropriate for the description of perturbations at the given stage of the evolution of the universe. Of course, it can be still used during the stages when the functions Φ , Ψ and δ are small in absolute value (before the solid component was formed as well as in the last period before the perturbation entered the horizon).

Newtonian gauge is usually viewed as closest to intuition, since the theory resembles Newton's theory of gravitation in it. However, this does not hold for superhorizon perturbations, which are substantially non-Newtonian since their properties do crucially depend on the finiteness of the speed of propagation of electromagnetic as well as gravitational interaction. As a result, even if there exists an intuitive explanation of the failure of Newtonian gauge in the first period after the shear stress has been switched, it is by no means self-evident. As it looks without further analysis, the failure is just a consequence of the peculiar character of coordinate transformations in which the new coordinates x^μ depend weakly on the old coordinates \mathbf{x} ($\delta x^\mu \propto e^{i\mathbf{k}\cdot\mathbf{x}}$ with $k\eta \ll 1$).

Even if the perturbations are small, the theory must be ruled out if it contradicts observations. This is where the restriction on the values of b used throughout this section comes from. Consider perturbations corresponding to the large-scale part of CMB anisotropies, $0.01k^{(0)} < k < k^{(0)}$. Observations suggest that the quantities Φ_{re} , Ψ_{re} and δ_{re} for such values of k are approximately constant. (The exact formulation of that is that the product of Fourier components of Φ_{re} , Ψ_{re} and δ_{re} as functions of \mathbf{x} , averaged over the ensemble of universes, is approximately proportional to $k^3\delta(\mathbf{k} - \mathbf{k}')$.) The quantities Φ_{re} , Ψ_{re} and δ_{re} are the values of the functions Φ , Ψ and δ at the moment $\zeta_{re} = (z_s^{(0)})^{-1} = k/k^{(0)} z_s^{-1}$. Thus, we require that the three functions are approximately constant throughout the interval $0.01z_s^{-1} < \zeta < z_s^{-1}$.

If $|b| \ll 1$, the functions Φ , Ψ and δ are varying slowly as long as they are dominated by the term proportional to ζ^{-b} . The first function passes to this regime at $\zeta \approx |b|z_s^{-1}$ and the other two functions pass to this regime at $\zeta \approx |b|^{1/2}z_s^{-1}$. This is confirmed by a more detailed analysis taking into account the presence of matter in cosmic medium. (In fact, even at $\zeta \ll \zeta_{re}$, when the

effect of matter is small, the exact expression for Φ looks different than in (38). It has the factor b in front of ζ^{-2-b} replaced by $b + \zeta/(2\zeta_{re})$. However, the correction starts to be important not earlier than at $\zeta \approx |b|\zeta_{re} = |b|z_s^{-1}$, which is just the moment when the term proportional to ζ^{-2-b} starts to be dominated by the term proportional to ζ^{-b} .) Since we require that all *three* functions are varying slowly, the relevant interval of ζ is the less of the two, $|b|^{1/2}z_s^{-1} \lesssim \zeta < z_s^{-1}$. In order that this is contained in the interval $0.01z_s^{-1} < \zeta < z_s^{-1}$, the parameter b must satisfy $|b| \lesssim 10^{-4}$. A natural description of shear stress is with the help of the parameter $\xi = w_+\beta$, equal to $b/6$ in case $w = 1/3$. The observational constraint on this parameter is $|\xi| \lesssim 10^{-5}$.

4 Stiff solid

4.1 Expansion of a universe with stiff solid

Suppose a solid with $w > 1/3$ appears in a universe filled with radiation. Density of matter decreases with the increasing scale parameter as a^{-3w_+} , the faster the greater the value of w . Thus, if the solid acquires a substantial part of the energy of radiation at the moment it is formed, it will dominate the evolution of the universe for a limited period until the radiation takes over again. Let us determine how the dynamics of such universe looks like.

Denote the part of the energy of radiation that transfers to the solid by $1 - \epsilon$. In the period with pure radiation ($\eta < \eta_s$) the mass density is $\rho = \rho_s(a_s/a)^4$, so that the first equation in (4) yields

$$a = C\eta, \quad C = \left(\frac{1}{6}\rho_s a_s^4\right)^{1/2}. \quad (45)$$

In the period with a mix of radiation and solid ($\eta > \eta_s$) the mass density is

$$\rho = \epsilon\rho_s(a_s/a)^4 + (1 - \epsilon)\rho_s(a_s/a)^{3w_+} = \rho_s(a_s/a)^4[\epsilon + (1 - \epsilon)(a_s/a)^\Delta],$$

where $\Delta = 3w_+ - 4$. As a result, the first equation in (4) transforms into

$$a' = C[\epsilon + (1 - \epsilon)(a_s/a)^\Delta]^{1/2}. \quad (46)$$

In the interval of w we are interested in the parameter Δ is positive, therefore the second term eventually becomes less than the first term even if $\epsilon \ll 1$.

Equation (46) solves analytically for $w = 2/3$ and $w = 1$, when $\Delta = 1$ and $\Delta = 2$. Note that for $w = 1$ it holds $\nu = 0$ and $n = \sqrt{-b}$; thus, in a universe filled with pure solid with $w = 1$, evolution of perturbations is smooth from the moment the solid was formed on only if the shear stress is negative. The solution is, for $w = 2/3$,

$$a = \frac{1 - \epsilon}{2\epsilon} a_s (\cosh \psi - 1), \quad C\tilde{\eta} = \frac{1 - \epsilon}{2\epsilon\sqrt{\epsilon}} a_s (\sinh \psi - \psi), \quad (47)$$

and for $w = 1$,

$$a = (\epsilon C^2 \tilde{\eta}^2 + 2a_s \sqrt{1 - \epsilon C \tilde{\eta}})^{1/2}. \quad (48)$$

In both formulas there appears *shifted time* $\tilde{\eta} = \eta - \eta_*$, with η_* defined in such a way that the resulting function $a(\eta)$ matches the function (45) at $\eta = \eta_s$. The shift is given for $w = 2/3$ by

$$\eta_* = \left[1 - \frac{1}{\epsilon} \left(1 - \frac{1 - \epsilon}{2\sqrt{\epsilon}} \log \frac{1 + \sqrt{\epsilon}}{1 - \sqrt{\epsilon}} \right) \right] \eta_s, \quad (49)$$

and for $w = 1$ by

$$\eta_* = \left(1 - \frac{1 - \sqrt{1 - \epsilon}}{\epsilon} \right) \eta_s. \quad (50)$$

In what follows we will use, instead of exact solutions for special w 's and any ϵ , approximate solution for any w and $\epsilon \ll 1$. Suppose less than one half of the total energy remains stored in radiation at the moment of radiation-to-solid transition ($\epsilon < 1/2$). The subsequent expansion of the universe can be divided into two eras, solid dominated and radiation dominated, separated by the time η_{rad} at which the mass densities of the solid and radiation are the same. The value of η_{rad} is given by

$$a_{rad} = a_s (\epsilon^{-1} - 1)^{1/\Delta}. \quad (51)$$

Suppose now that the post-transitional share of energy stored in radiation is small ($\epsilon \ll 1$). The universe then expands by a large factor between the times η_s and η_{rad} ,

$$a_{rad} \doteq a_s \epsilon^{-1/\Delta} \gg a_s,$$

and we can describe it in a good approximation as if it was filled first with pure solid and then with pure radiation. Thus, we replace equation (46) by

$$a' \doteq \begin{cases} C(a_s/a)^{\Delta/2} & \text{for } \eta < \eta_{rad} \\ \sqrt{\epsilon} C & \text{for } \eta > \eta_{rad} \end{cases}. \quad (52)$$

The solution is

$$a \doteq \begin{cases} [(\Delta/2 + 1) a_s^{\Delta/2} C \tilde{\eta}]^{\frac{1}{\Delta/2+1}} & \text{for } \eta < \eta_{rad} \\ \sqrt{\epsilon} C \tilde{\eta} & \text{for } \eta > \eta_{rad} \end{cases}, \quad (53)$$

where $\tilde{\eta}$ and $\tilde{\tilde{\eta}}$ are shifted time variables, $\tilde{\eta} = \eta - \eta_*$ and $\tilde{\tilde{\eta}} = \tilde{\eta} - \eta_{**}$. From the approximate expression for a_{rad} we obtain

$$\tilde{\eta}_{rad} = \frac{1}{\Delta/2 + 1} \epsilon^{-\frac{\Delta/2+1}{\Delta}} \eta_s, \quad (54)$$

and by matching the solutions at η_s and η_{rad} we find

$$\eta_* = \frac{\Delta/2}{\Delta/2 + 1} \eta_s, \quad \eta_{**} = -\frac{\Delta}{2} \tilde{\eta}_{rad}, \quad (55)$$

As a quick test of the exact solutions cited above we can check that expressions (49) and (50) for η_* reduce to the first expression in (55) in the limit $\epsilon \ll 1$.

The two equations in (55) can be rewritten as

$$\frac{\tilde{\eta}_s}{\eta_s} = \frac{1}{\Delta/2 + 1} = \frac{u}{u_0}, \quad \frac{\tilde{\eta}_{rad}}{\eta_{rad}} = \frac{\Delta}{2} + 1 = \frac{u_0}{u},$$

where u_0 is the value of u in the radiation era. (We have used that $u = 1/(\Delta + 2)$ and $u_0 = 1/2$.) Expressions for the ratios $\tilde{\eta}_s/\eta_s$ and $\tilde{\eta}_{rad}/\eta_{rad}$ in terms of the ratio u/u_0 stay valid also after we replace radiation by an ideal fluid with arbitrary pressure-to-radiation ratio w_0 . To demonstrate that, let us derive them from the condition of continuity of Hubble parameter. If in the given period of time the universe is filled with matter with the given value of w , its scale parameter depends on a suitably shifted time $\tilde{\eta}$ as $a \propto \tilde{\eta}^{2u}$. Thus, its Hubble parameter is $\mathcal{H} = 2u\tilde{\eta}^{-1}$ and the requirement that \mathcal{H} is continuous at the moment when w changes from w_I to w_{II} is equivalent to $\tilde{\eta}_{II}/\tilde{\eta}_I = u_{II}/u_I$.

4.2 Transitions with jump in w

Suppose the functions w_η and β_η change at the given moment η_{tr} (“transition time”) from (w_I, β_I) to $(w_{II}, \beta_{II}) = (w_I + \Delta w, \beta_I + \Delta\beta)$. (We have attached the index η to the symbols w and β in order to distinguish the functions denoted by them from the values these functions assume in a particular era.) Rewrite the first equation in (11) as

$$\mathcal{B}' = c_{S0}^2(3\mathcal{H}\mathcal{B} + \mathcal{E}) + \left(\frac{3}{2}w_{\eta+} - 1\right)\mathcal{H}\mathcal{B} + \frac{4}{3}\beta_\eta\mathcal{E}, \quad (56)$$

where

$$c_{S0}^2 = \frac{dp}{d\rho} = w_\eta + \rho \frac{dw_\eta}{d\rho}. \quad (57)$$

Because of the jump in w_η there appears δ -function in c_{S0}^2 , and to account for it, we must assume that \mathcal{B} has a jump, too. However, on the right hand side of equation (56) we then obtain an expression of the form “ θ -function \times δ -function”; and if we rewrite \mathcal{B}' as

$$\mathcal{B}' = \frac{d\mathcal{B}}{d\rho}\rho' = -3\mathcal{H}\rho w_{\eta+} \frac{d\mathcal{B}}{d\rho},$$

on the left hand side there appears another such expression. To give meaning to the equation we must suppose that w_η changes from w_I to w_{II} within an interval of the length $\Delta\rho \ll \rho_{tr}$, and send $\Delta\rho$ to zero in the end. If we retain just the leading terms in equation (56) in the interval under consideration, we obtain

$$w_{\eta+} \frac{d\mathcal{B}}{d\rho} = -\left(\mathcal{B} + \frac{\mathcal{E}_{tr}}{3\mathcal{H}_{tr}}\right) \frac{dw_\eta}{d\rho}, \quad (58)$$

where we have used the fact that, as seen from the second equation in (11), the function \mathcal{E} is continuous at $\eta = \eta_{tr}$. The solution is

$$\mathcal{B} + \frac{\mathcal{E}_{tr}}{3\mathcal{H}_{tr}} = \frac{\mathcal{C}}{w_{\eta+}}.$$

To compute the jump in \mathcal{B} , we express \mathcal{B}_I and \mathcal{B}_{II} in terms of w_{I+} and w_{II+} , compute the difference $\mathcal{B}_{II} - \mathcal{B}_I$ and use the expression for \mathcal{B}_I to exclude \mathcal{C} . In this way we find

$$[\mathcal{B}] = -\frac{\Delta w}{w_{II+}} \left(\mathcal{B}_I + \frac{\mathcal{E}_{tr}}{3\mathcal{H}_{tr}} \right). \quad (59)$$

Note that the same formula is obtained if we assume that the functions with jump are equal to the mean of their limits from the left and from the right at the point where the jump occurs.

To justify the expression for $[\mathcal{B}]$ we can compute the jump in Ψ ,

$$[\Psi] = -\frac{3}{2}k^{-2}\mathcal{H}_{tr}^2(3\mathcal{H}_{tr}[w_{\eta+}\mathcal{B}] + \Delta w\mathcal{E}_{tr}).$$

If we write $[w_{\eta+}\mathcal{B}] = w_{II+}[\mathcal{B}] + \Delta w\mathcal{B}_I$ and insert for $[\mathcal{B}]$, we immediately see that $[\Psi]$ vanishes. This must be so because a jump in Ψ would produce a derivative of δ -function in equation (36), and no such expression with opposite sign appears in the other terms present there.

The jump in \mathcal{B}' can be found from equation (56) by computing the jump of the right hand side, with no need for the limiting procedure we have used when determining the jump in \mathcal{B} . The result is

$$[\mathcal{B}'] = 4\frac{\Delta w}{w_{II+}}\mathcal{H}_{tr}\mathcal{B}_{tr} + \left(\frac{5-3w_{II}}{6w_{II+}}\Delta w + \frac{4}{3}\Delta\beta \right)\mathcal{E}_{tr}. \quad (60)$$

4.3 Perturbations in a universe with stiff solid

We are interested in perturbations in a universe in which the parameters w and β assume values $(w_0, 0)$ before η_s , (w, β) between η_s and η_{rad} , and $(w_0, 0)$ after η_{rad} . (For most of this subsection we will leave w_0 free, only at the end we will put $w_0 = 1/3$.) Denote the functions describing the perturbation before η_s by the index 0, between η_s and η_{rad} by the index s , and after η_{rad} by the index 1. If only the nondecaying part of perturbation survives before the moment when the solid appears, \mathcal{B}_0 and \mathcal{E}_0 are given by expressions (28) with u replaced by u_0 and q_0 defined as $\sqrt{w_0}k$. If, furthermore, the perturbation is stretched far beyond the horizon all the time, \mathcal{B}_s and \mathcal{E}_s are given by expressions (21) and (23) with z replaced by $\tilde{z} = q\tilde{\eta}$, and \mathcal{B}_1 is given by expression (24) with c_J and c_Y replaced by c_{J1} and c_{Y1} , J , Y and ν replaced by J_0 , Y_0 and ν_0 , and z replaced by $\tilde{\tilde{z}} = q_0\tilde{\tilde{\eta}}$. All we need to obtain the complete description of the perturbation is to match expressions for \mathcal{B}_0 , \mathcal{B}_s and \mathcal{B}_1 with the help of expressions for \mathcal{E}_0 and \mathcal{E}_s at the moments η_s and η_{rad} .

At the moment η_s , the jumps in w_η and β_η are $\Delta w_s = w - w_0 \equiv \Delta w$ and $\Delta\beta_s = \beta$. By using these values and the identity $\mathcal{E}_0 = -3\mathcal{H}_s\mathcal{B}_{0s}$ we find

$$[\mathcal{B}]_s = 0, \quad [\mathcal{B}']_s = -\left(\frac{1}{2}\Delta w - \frac{4}{3}\beta \right)\mathcal{E}_0,$$

The resulting equations for the unknowns $\tilde{x} = c_J J \tilde{z}_s^{-m}$ and $\tilde{y} = c_Y Y \tilde{z}_s^{-M}$ are

$$\tilde{x} + \tilde{y} = C \frac{u_0}{u}, \quad (1-m)\tilde{x} + (1-M)\tilde{y} = C \left[1 - \left(\beta - \frac{3}{8}\Delta w \right) 8u_0 \right], \quad (61)$$

and their solution is

$$\tilde{x} = C \frac{u_0}{u} \frac{1}{2n} (M - \sigma_2), \quad \tilde{y} = -C \frac{u_0}{u} \frac{1}{2n} (m - \sigma_2). \quad (62)$$

Potentials Φ and Ψ , computed from the functions \mathcal{B} and \mathcal{E} , are enhanced by a factor of order $\beta \tilde{z}_s^{-2} \Phi_0$ for η close to η_s . The former potential jumps either up or down by the value of that order, while the latter potential rises or falls abruptly without a jump. (This follows from the fact that \tilde{x} and \tilde{y} are proportional to $M - \sigma_2$ and $-(m - \sigma_2)$, just as x and y computed earlier for a universe in which the parameter w did not change during the fluid-to-solid transition.) As before, exploding Φ and Ψ do not disrupt the theory, since \mathcal{B} and \mathcal{E} vary smoothly enough.

At the moment η_{rad} , the jumps in w_η and β_η are $\Delta w_{rad} = -\Delta w$ and $\Delta \beta_{rad} = -\beta$. By inserting these values into the expressions for $[\mathcal{B}]$ and $[\mathcal{B}']$ we obtain

$$[\mathcal{B}]_{rad} = \frac{\Delta w}{w_{0+}} \left(\mathcal{B}_{s,rad} + \frac{\mathcal{E}_{rad}}{3\mathcal{H}_{rad}} \right), \quad [\mathcal{B}']_{rad} = -4 \frac{\Delta w}{w_{0+}} \mathcal{H}_{rad} \mathcal{B}_{s,rad} - \left(\frac{5-3w_0}{6w_{0+}} \Delta w + \frac{4}{3} \beta \right) \mathcal{E}_{rad}.$$

(\mathcal{B}_s refers to the function \mathcal{B} between the times η_s and η_{rad} , hence $\mathcal{B}_{s,rad}$ is the limit of that function for η approaching η_{rad} from the left. The quantity \mathcal{E}_{rad} is to be understood in the same way.) Introduce the variables

$$\tilde{X} = c_J J \tilde{z}_{rad}^{-m} = p^{-m} \tilde{x}, \quad \tilde{Y} = c_Y Y \tilde{z}_{rad}^{-M} = p^{-M} \tilde{y}, \quad (63)$$

where p is the ratio of final and initial moments of the period during which the solid affects the dynamics of the universe, $p = \tilde{\eta}_{rad}/\tilde{\eta}_s$. Equations for the unknowns $\tilde{x} = c_{J1} J_0$ and $\tilde{y} = c_{Y1} Y_0 \tilde{z}_{rad}^{-2\nu_0}$ are

$$\tilde{x} + \tilde{y} = \frac{q}{q_0} \frac{u}{u_0} (K_J \tilde{X} + K_Y \tilde{Y}), \quad \tilde{x} + (1 - 2\nu_0) \tilde{y} = \frac{q}{q_0} (L_J \tilde{X} + L_Y \tilde{Y}), \quad (64)$$

where the coefficients on the right hand side are defined as

$$K_J = \frac{1}{w_{0+}} \left[w_+ - \frac{\Delta w}{6u\tilde{w}} (m + \tau) \right], \quad K_Y = \text{ditto with } m \rightarrow M,$$

and

$$L_J = 1 - m - \frac{8u\Delta w}{w_{0+}} + \frac{m + \tau}{\tilde{w}} \left(\frac{5-3w_0}{6w_{0+}} \Delta w + \frac{4}{3} \beta \right), \quad L_Y = \text{ditto with } m \rightarrow M,$$

The solution is

$$\tilde{x} = \frac{1}{2\nu_0} \frac{q}{q_0} (M_J \tilde{X} + M_Y \tilde{Y}), \quad \tilde{y} = -\frac{1}{2\nu_0} \frac{q}{q_0} (N_J \tilde{X} + N_Y \tilde{Y}) \quad (65)$$

with the constants M_α and N_α , $\alpha = J, Y$, defined in terms of the constants L_α and K_α as

$$M_\alpha = L_\alpha - (1 - 2\nu_0) \frac{u}{u_0} K_\alpha, \quad N_\alpha = L_\alpha - \frac{u}{u_0} K_\alpha.$$

The nondecaying part of the function Φ in the period after the radiation takes over again is

$$\Phi_1 = C_{J1} J_{0+} = \frac{\Sigma_0 q_0}{2\nu_{0+}} \tilde{x}.$$

Here we must insert for \tilde{x} from equation (65), with \tilde{X} and \tilde{Y} given in equation (63) and \tilde{x} and \tilde{y} given in equation (62). The constant C in the latter equation is given by the expression following from equation (31) with Σ replaced by Σ_0 and ν replaced by ν_0 ,

$$C = \frac{2\nu_{0+}}{\Sigma_0 q} \Phi_0.$$

The resulting expression for Φ_1 is

$$\Phi_1 = \frac{1}{2\nu_0} \frac{u_0}{u} \frac{1}{2n} (\hat{M}_J p^{-m} - \hat{M}_Y p^{-M}) \Phi_0, \quad (66)$$

with the coefficients \hat{M}_J and \hat{M}_Y defined as

$$\hat{M}_J = M_J(M - \sigma_2), \quad \hat{M}_Y = M_Y(m - \sigma_2).$$

After some algebra the coefficients reduce to

$$\hat{M}_J = 2\nu_0 \frac{u}{u_0} M - b, \quad \hat{M}_Y = \text{ditto with } M \rightarrow m. \quad (67)$$

In a universe filled with ideal fluid, the potentials Φ and Ψ coincide and are continuous together with their derivative at the moment when w jumps to the new value. Thus, if Φ did not contain the decaying term at the beginning, it does not develop it during the jump. As a result, its value stays the same. (This is not true if w changes continuously. For example, during the radiation-to-matter transition shortly before recombination Φ decreases by the factor 9/10.) In our problem with $\beta = 0$, the medium filling the universe is ideal fluid whose parameter w changes abruptly from one value to another and back again, therefore Φ_1 (final value of Φ) equals Φ_0 (initial value of Φ). The same result is obtained from equations (66) and (67), if we insert $n = \nu$, $M = 2\nu$ and $m = b = 0$ into them.

Let us now determine how fast the function Φ approaches its limit value. The decaying part of Φ in the period under consideration is

$$\Delta\Phi_1 = -2\nu_{0+} \frac{u_0}{u} \frac{1}{2n} (\hat{N}_J p^{-m} - \hat{N}_Y p^{-M}) \tilde{z}_{rad}^{-2} \zeta^{-2\nu_{0+}} \Phi_0, \quad (68)$$

where ζ is rescaled time normalized to 1 at the moment η_{rad} , $\zeta = \tilde{z}/\tilde{z}_{rad}$, and the coefficients \hat{N}_J and \hat{N}_Y are defined in terms of N_J and N_Y in the same way as the coefficients \hat{M}_J and \hat{M}_Y in terms of M_J and M_Y . After rewriting the former coefficients similarly as we did with the latter ones, we obtain

$$\hat{N}_J = \hat{N}_Y = -\frac{w_0}{w_{0+}} 2b. \quad (69)$$

From these equations and equations (66) and (67) we find that the ratio of the decaying and nondecaying part of Ψ at the moment of solid-to-radiation transition is

$$\frac{\Delta\Phi}{\Phi_1} \Big|_{rad} = R_{rad} \tilde{z}_{rad}^{-2}, \quad R_{rad} = 4\nu_0 \nu_{0+} \frac{w_0}{w_{0+}} \frac{2u_0 b}{2\nu_0 u [n \coth(n \log p) + \nu] - u_0 b}. \quad (70)$$

The ratio is greater than one for $|\beta| \gtrsim \tilde{z}_{rad}^2$. The function Φ is then dominated by the decaying term at the moment η_{rad} , and the nondecaying term takes over later, at the moment η_{nd} given by

$$\tilde{z}_{nd} = R_{rad}^{\frac{1}{2\nu_0+}} \tilde{z}_{rad}^{1-\frac{1}{\nu_0+}}. \quad (71)$$

The exponent at \tilde{z}_{rad} is positive for any $w_0 < 1$ (it equals $1/3$ for $w_0 = 1/3$) and the constant R_{rad} is of order 1 or less. Thus, if the perturbation was large-scale at the moment the fluid originally filling the universe started to be dominating again ($\tilde{z}_{rad} \ll 1$), it will be still large-scale at the moment the nondecaying term prevails over the decaying one ($\tilde{z}_{nd} \ll 1$).

The time η_{rad} must not be too close to the time of recombination, if the spectrum of large-size CMB anisotropies is not to be distorted. Suppose $w_0 = 1/3$ and denote the values of the field $\Phi_{1tot} = \Phi_1 + \Delta\Phi$ which it assumes at the moment η_{re} for wave numbers $k^{(0)}$ and $0.01k^{(0)}$ by $\Phi^{(0)}$ and $\Phi^{(1)}$. Their ratio is

$$\frac{\Phi^{(0)}}{\Phi^{(1)}} = \frac{1 + R_{rad}\tilde{z}_{rad}^{(0)}}{1 + 10^4 R_{rad}\tilde{z}_{rad}^{(0)}} \doteq 1 - 10^4 R_{rad}\tilde{z}_{rad}^{(0)}.$$

The expression on the left hand side equals 0.01^{n_S-1} , where n_S is the spectral index, whose deviation from 1 (about -0.04 according to observations) describes the tilt of the scalar spectrum. If we allow for a tilt of the primordial spectrum, too, the right hand side will be multiplied by $0.01^{n_{S0}-1}$. Denote $p_* = 1/\tilde{z}_{rad}^{(0)} = \tilde{\eta}_{re}/\tilde{\eta}_{rad} = a_{re}/a_{rad} = T_{rad}/T_{re}$ and require that n_{S0} differs from n_S at most by some $\Delta n_S \ll 1$. To ensure that, p_* must satisfy

$$p_* > 2 \times 10^3 R_{rad} \Delta n_S^{-1}. \quad (72)$$

For numerical calculations we need the value of p . It is a ratio of *times*, but can be rewritten in terms of a ratio of *scale parameters* or *temperatures*, $P = a_{rad}/a_s = T_s/T_{rad}$, as

$$p = P^{\frac{1}{2u}}. \quad (73)$$

The value of p , or equivalently, P , determines the interval of admissible w 's. To obtain it, note that for $w_0 = 1/3$ equation (51) yields $P = (\epsilon^{-1} - 1)^{1/\Delta} \doteq \epsilon^{-1/\Delta}$, or

$$P \doteq \epsilon^{-\frac{1}{3\Delta w}}. \quad (74)$$

(This is consistent with equation (54), which can be rewritten as $p = \epsilon^{-\frac{\Delta/2+1}{\Delta}} = \epsilon^{-\frac{1}{6u\Delta w}}$.) Thus, the jump in the parameter w for the given ratio P must satisfy

$$\Delta w \doteq \frac{\log 1/\epsilon}{3 \log P} \gtrsim \frac{1}{3 \log P}. \quad (75)$$

The dependence of the quantities $\tilde{\Phi}_1 = \Phi_1/\Phi_0$ and R_{rad} on the parameter β is depicted in fig. 2. The values of w_0 and w are $1/3$ and $2/3$ on both panels, and the solid and dotted lines

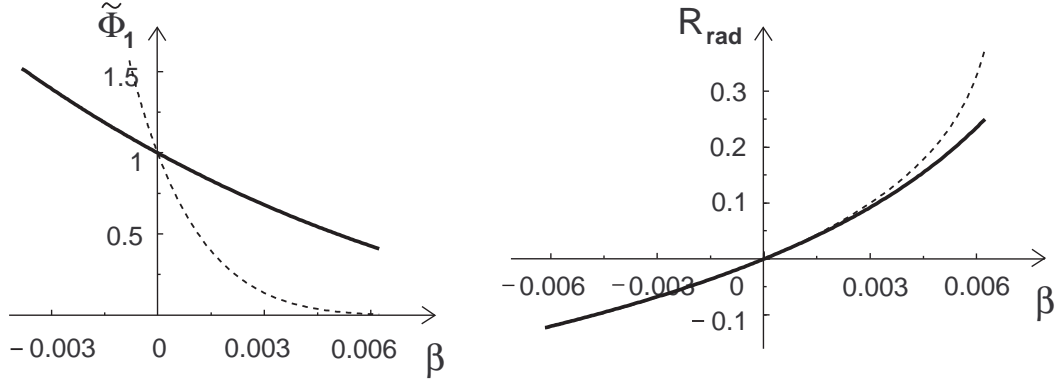


Fig. 2: Final value of Newtonian potential in a universe with stiff solid (left) and normalized ratio of decaying to nondecaying part of the potential at solid-to-radiation transition (right), plotted as functions of shear modulus

correspond to $P = 10^3$ and $P = 10^{13}$ respectively. The lines are terminated at $\beta = 1/160$, which is the maximum admissible β for $w = 2/3$.

The parameter P assumes the smaller value if, for example, the solid dominated the dynamics of the universe between the electroweak and confinement scale, and the greater value, if the solid was formed as soon as at the GUT scale and dominated the dynamics of the universe up to the electroweak scale. Unless the parameter w of the solid is close to that of radiation, the fraction of energy which remains stored in radiation after the solid has been formed must be quite small in the former case and very small in the latter case. For $w = 2/3$ this fraction equals $1/P$, so that for the greater P the mechanism of the radiation-to-solid transition must transfer to the solid all but one part in 10 trillions of the energy of radiation.

The quantity $\tilde{\Phi}_1$ is the factor by which the function Φ changes due to the presence of stiff solid in the early universe. From the figure we can see that Φ is shifted upwards for $\beta < 0$ and downwards for $\beta > 0$, and the enhancement factor decreases monotonically with β , the steeper the larger the value of P . For maximum β the function Φ is suppressed by the factor 0.41 if $P = 10^3$ and by the factor 0.004 if $P = 10^{13}$.

The quantity R_{rad} determines, together with Δn_S , the minimal duration of the period between the moment when radiation took over again and recombination. From the figure we can see that for maximum β and $P = 10^3$ the temperature at the beginning of this period had to be at least $8 \times 10^3 \Delta n_S^{-1} T_{re} \doteq 0.2 (\Delta n_S / 0.01)^{-1}$ MeV.

Finally, let us compute the quantities Φ_1 and R_{rad} for maximum β as functions of w . The maximum β corresponds to $n = 0$ and equals

$$\beta_m = \frac{3}{32} \frac{(1-w)^2}{w_+}. \quad (76)$$

If we perform the limit $\beta \rightarrow \beta_m$ in the expression for Φ_1 , we obtain

$$\Phi_{1m} = \frac{1}{2\nu_0} \frac{u_0}{u} (\hat{M}_{0m} \log p + \hat{M}_1) p^{-\nu} \Phi_0, \quad (77)$$

where \hat{M}_0 and \hat{M}_1 are the first two coefficients in the expansion of \hat{M}_J in the powers of n ,

$$\hat{M}_0 = 2\nu_0 \frac{u}{u_0} \nu - b, \quad \hat{M}_1 = 2\nu_0 \frac{u}{u_0}.$$

The quantity R_{rad} computed in the limit $\beta \rightarrow \beta_m$ is

$$R_{rad,m} = 4\nu_0 \nu_{0+} \frac{w_0}{w_{0+}} \frac{2u_0 b_m}{2\nu_0 u (1/\log p + \nu) - u_0 b_m}. \quad (78)$$

For $w = 1$ it holds $\beta_m = 0$ and hence $\Phi_{1m} = \Phi_0$ and $R_{rad,m} = 0$. As w decreases, Φ_{1m} decreases, too, and it reaches minimum for the value of Δw given on the right hand side of equation (75). For $P = 10^{13}$, the minimum is as small as 3×10^{-6} . The quantity $R_{rad,m}$, on the other hand, decreases with *increasing* w ; for example, it falls from 0.82 to 0 for $P = 10^3$ and from 1.28 to 0 for $P = 10^{13}$, as w increases from the minimum value given in equation (75) to 1. (All numerical values refer to $w_0 = 1/3$.)

5 Conclusion

We have studied the effect of solid matter with $w > 0$ on the evolution of scalar perturbations. Two scenarios were analyzed: a scenario with radiation-like solid appearing in the universe at some moment during the radiation era and staying there till recombination, and a scenario with stiff solid appearing also in the radiation era and dominating the evolution of the universe during a limited period before recombination. The focus was on long-wavelength (supercurvature) perturbations, therefore it was necessary to assume that the solidification was anisotropic, producing a solid with flat internal geometry; otherwise no new effect would be obtained. In the calculations, proper-time comoving gauge was used instead of more common, and intuitively more appealing, Newtonian gauge. Besides being more convenient computationally, the gauge turned out to be preferable also on principal grounds, since the requirement of smallness of perturbations was not violated in it. (Note that the description of long-wavelength perturbations in Newtonian gauge breaks down also in case $w < 0$, because evolutionary equations are then numerically unstable. This forced the authors of [1] to switch to *synchronous gauge* after formulating the theory in Newtonian gauge.) Any theory of long-wavelength perturbations must be consistent with the observational fact that their spectrum at recombination is flat. We have shown that in the problem with radiation-like solid this leads to a constraint on ξ (dimensionless shear modulus), and in the problem with stiff solid this yields a constraint on the ratio of T_{rad} (temperature at solid-to-radiation transition) to T_{re} (temperature at recombination): in order that the theory does not contradict observations,

$|\xi|$ must be small enough and T_{rad}/T_{re} must be large enough. The net effect of stiff solid is suppression of Newtonian potential in case $\xi > 0$ and enhancement of it in case $\xi < 0$. This might raise hope that for $\xi > 0$ also the scalar-to-tensor ratio is suppressed, which would surely be an interesting effect from the observational point of view. However, a straightforward calculation shows that tensor perturbations are suppressed by exactly the same factor as scalar ones. Thus, the ratio would change only if its *initial value* would change; and whether this happens or not depends on how the inflationary scenario is modified to provide appropriate initial conditions for scalar perturbations.

A Relativity

Consider an elastic medium put into the metric $g_{\mu\nu}$, whose body coordinates X^A are given functions of the spacetime coordinates x^μ . The deformation of the medium is described by the body metric H^{AB} , defined as the spacetime metric push-forwarded to the body space,

$$H^{AB} = -g^{\mu\nu} X^A_{,\mu} X^B_{,\nu}. \quad (\text{A-1})$$

Material properties of the medium are encoded in the constitutive equation $\rho = \rho(H^{AB})$. Knowing the function $\rho(H^{AB})$, one computes the energy-momentum tensor as

$$T_{\mu\nu} = 2 \frac{\partial \rho}{\partial H^{AB}} X^A_{,\mu} X^B_{,\nu} + \rho g_{\mu\nu}. \quad (\text{A-2})$$

A special kind of elastic medium is an ideal fluid. To define it, one introduces particle density n . In general, n is proportional to the particle density n_{ref} which would be observed in the medium if transformed into some properly chosen reference relaxed state. If the space is filled with one kind of medium only, the density n_{ref} can be rescaled to 1 and the actual particle density can be written as

$$n = (\det H^{AB})^{1/2}. \quad (\text{A-3})$$

By definition, ideal fluid is a medium whose energy density depends on H^{AB} only through n .

An important new concept in relativity is that of the *partially relaxed state*, defined as the state in which the medium has minimum energy per particle $\epsilon = \rho/n$ at fixed n . Consider a state close to the partially relaxed state and write the quantities appearing in equations (A-1), (A-2) and (A-3) as $f = f^{(0)} + \delta f$, where $f^{(0)}$ is the value in the partially relaxed state and δf is a small corrections to it. If the medium is isotropic, the constitutive equation reads

$$\delta \epsilon = -\sigma^{(0)} \frac{\delta V}{V^{(0)}} + \frac{1}{8} \bar{\lambda} (\delta H_A^A)^2 + \frac{1}{4} \bar{\mu} (\delta H_A^B)^2, \quad (\text{A-4})$$

where σ is pressure energy per particle, $\sigma = p/n$, $\bar{\lambda}$ and $\bar{\mu}$ are Lamé coefficients per particle, $\bar{\lambda} = \lambda/n$ and $\bar{\mu} = \mu/n$, V is volume per particle, $V = 1/n$, and the first index of the tensor δH^{AB}

is lowered by the matrix $H_{AB}^{(0)}$ inverse to the matrix $H^{(0)AB}$, $\delta H_A^B = H_{AC}^{(0)} \delta H^{CB}$. In the last term, the “implicit summation rule” is used, $(\delta H_A^B)^2 = \delta H_A^B \delta H_B^A$. To compute $T_{\mu\nu}$, we need to express $\delta\epsilon$ in terms of δH_A^B only. This is achieved by writing the ratio $\delta V/V^{(0)}$ on the right hand side of (A-4) as

$$\frac{\delta V}{V^{(0)}} = \frac{1}{2} \delta H_A^A + \frac{1}{2} (\delta H_A^A)^2 + \frac{1}{4} (\delta H_A^B)^2.$$

Equation (A-4) holds to the second order in δH_A^B . Within this accuracy, the trace δH_A^A in the λ -term can be replaced by $2\delta V/V$, so that if μ vanishes, $\delta\epsilon$ as well as $\delta\rho$ depends on δH only through δV and we are dealing with an ideal fluid. Note also that by comparing (A-4) to the Taylor expansion of $\epsilon(V^{(0)} + \delta V)$, passing from ϵ and σ to ρ and p and skipping the index (0), one obtains

$$\frac{d\rho}{dV} = -\frac{\rho_+}{V}, \quad \frac{dp}{dV} = -\frac{K}{V}, \quad (\text{A-5})$$

where ρ_+ and K are defined in section 2. If we introduce an auxiliary sound speed c_{S0} defined in terms of the function $p(\rho)$ in the same way as the sound speed of an ideal fluid, $c_{S0}^2 = dp/d\rho$, we find

$$c_{S0}^2 = \frac{K}{\rho_+}. \quad (\text{A-6})$$

In an unperturbed universe, the 3-space coordinates are comoving, $\mathbf{x} = \mathbf{X}$, and the matter at any given moment is in a partially relaxed state with $H^{(0)ij} = -g^{ij} = a^{-2}\delta_{ij}$. In a perturbed universe, the 3-space coordinates differ from the body coordinates by a small displacement vector $\boldsymbol{\xi}$, $\mathbf{x} = \mathbf{X} + \boldsymbol{\xi}$, and the body metric acquires a small correction δH^{ij} . This yields

$$T_0^0 = \rho^{(0)} - \frac{1}{2}\rho_+ \delta H_k^k, \quad T_i^0 = \rho_+ (-\xi^{i'} + h_{0i}), \quad T_i^j = -p^{(0)}\delta_{ij} + \frac{1}{2}\lambda \delta H_k^k \delta_{ij} + \mu \delta H_i^j. \quad (\text{A-7})$$

With the index (0) at ρ and p skipped, these expressions reduce to those cited in section 2. Let us verify that. In an interval of conformal time $d\eta$, the proper time τ of any given volume element of the medium increases approximately by $d\tau = a d\eta$, so that $\mathbf{u} \equiv d\mathbf{x}/d\tau = a^{-1}\boldsymbol{\xi}'$, $u_i = a(-\xi^{i'} + h_{0i})$ and expression for T_i^0 coincides with that in equation (2). By comparing the expressions for T_0^0 and T_i^j with those in extended equation (2) we obtain

$$\delta\rho = -\frac{1}{2}\rho_+ \delta H_k^k, \quad \delta p = -\frac{1}{2}K \delta H_k^k, \quad \Delta T_i^j = \mu \delta \tilde{H}_i^j, \quad (\text{A-8})$$

where the tilde denotes the traceless part of the matrix, $\delta \tilde{H}_i^j = \delta H_i^j - \frac{1}{3}\delta H_k^k \delta_{ij}$. From the definition of H^{AB} it follows

$$\delta H_i^j = \xi^i_{,j} + \xi^j_{,i} - h_{ij}, \quad (\text{A-9})$$

so that in the comoving gauge used throughout the paper, in which $\xi^i = 0$, we have $\delta H_i^j = -h_{ij}$. After inserting this into equation (A-8) and using the definition of $\delta\tau_{ij}$, we arrive at the expressions for $\delta\rho$ and $\delta\tau_{ij}$ in equation (3). Note also that from the first two equations it follows that the ratio $\delta p/\delta\rho$ equals the derivative $dp/d\rho$.

Equation (A-9) holds only for elastic media with *flat internal geometry*. In case the internal geometry is perturbed, we must distinguish between local body coordinates X^i and global body coordinates \mathcal{X}^i , and write the body metric tensor in the latter coordinates as

$$\mathcal{H}^{ij} \equiv \frac{\partial \mathcal{X}^i}{\partial X^k} \frac{\partial \mathcal{X}^j}{\partial X^l} H^{kl} = a^{-2}(\delta_{ij} + b_{ij}).$$

The formulas (A-8) then remain valid, but in the formula (A-9) there appears an extra term

$$\Delta H_i{}^j = -b_{ij}. \quad (\text{A-10})$$

When using the expression (A-4) for $\delta\epsilon$ in a perturbed universe, we have tacitly assumed that the entropy per particle S is constant throughout the space. Thus, we have considered *adiabatic perturbations* only. More general are *entropy perturbations* which include nonzero correction to the entropy per particle δS . For such perturbations, there appear additional terms proportional to δS in the first two formulas in (A-8). The explicit form of these terms is

$$\Delta\rho = nT\delta S, \quad \Delta p = n^2 \left(\frac{\partial T}{\partial n} \right)_S \delta S, \quad (\text{A-11})$$

where T is the temperature of the medium.

Consider a perturbation in the form of plane wave with the given wave vector \mathbf{k} , and suppose it is located well inside the horizon, $k\eta \gg 1$. For such perturbation, longitudinal and transverse sound speeds $c_{S\parallel}$ and $c_{S\perp}$ are given within a good accuracy by the same formulas as in Minkowski space,

$$c_{S\parallel}^2 = \frac{\lambda + 2\mu}{\rho_+}, \quad c_{S\perp}^2 = \frac{\mu}{\rho_+}. \quad (\text{A-12})$$

Corrections to these expressions are of order $(k\eta)^{-2}$.

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